A NEW INTERPRETATION OF BETHE ANSATZ SOLUTIONS FOR MASSIVE THIRRING MODEL

T. FUJITA[1] , Y. SEKIGUCHI and K. YAMAMOTO

Department of Physics, Faculty of Science and Technology

Nihon University, Tokyo, Japan

ABSTRACT

We reexamine Bethe ansatz solutions of the massive Thirring model. We solve equations of periodic boundary conditions numerically without referring to the density of states. It is found that there is only one bound state in the massive Thirring model. The bound state spectrum obtained here is consistent with Fujita-Ogura's solutions of the infinite momentum frame prescription. Further, it turns out that there exist no solutions for *string*−like configurations. Instead, we find boson boson scattering states in 2−particle 2−hole configurations where all the rapidity variables turn out to be real.

Manuscript pages 36: Figures 6: Table 4

[1]e-mail: fffujita@phys.cst.nihon-u.ac.jp

Proposed running head: " Bethe ansatz for massive Thirring"

Mailing Address:

T. Fujita

Department of Physics

Faculty of Science and Technology

Nihon University

Kanda-Surugadai

Tokyo, Japan

Telephone: (03)3259-0887 (Japan)

Fax: (03)3259-0887 (Japan)

e-mail: fffujita@phys.cst.nihon-u.ac.jp

1. Introduction

The sine-Gordon field theory or the massive Thirring model is believed to be solved exactly. In their classic paper, Dashen, Hasslacher and Neveu presented their solutions to the quantum sine-Gordon model [1]. Although they use semiclassical approximations, they consider their solutions to be *exact*. This is because their solutions seem to have

proper weak and strong coupling limits. Also, they showed that the sine-Gordon model has a rich spectrum of the charge zero sector. This spectrum is translated into the massive Thirring model and the bound state mass $\mathcal{M}$ (vector boson) is written as

$$\mathcal{M} = 2m \sin \frac{\pi}{2} \frac{n}{(1 + \frac{2g_0}{\pi})} \tag{1.1}$$

where n is an integer and runs from 1 to $(1 + \frac{2g_0}{\pi})$. m is the fermion mass of the massive Thirring model. g_0 is the coupling constant with Schwinger's normalization.

Further, this spectrum is confirmed by the Bethe ansatz solution [2]. This was very important since the Bethe ansatz wave function is indeed exact. In their paper, Bergknoff and Thacker presented their solutions of the massive Thirring model based on the *string* hypothesis when they solve equations of the periodic boundary conditions (PBC) from Bethe ansatz wave functions.

In this way, the massive Thirring model has been considered to be solved exactly and is supposed to possess many bound states.

However, Fujita and Ogura [3] have recently presented their solutions of the massive Thirring model employing infinite momentum frame prescription. Their spectrum is quite different from eq. (1.1). There is only one bound state. However, the bound state energy is rather close to the lowest energy of eq.(1.1). The deviation is about $10 \sim 20$ %

from each other depending on the coupling constant. The boson mass $\mathcal{M}$ is given as

$$\frac{\tan\alpha}{\frac{\pi}{2}-\alpha} = \frac{g}{\pi}\left[1+\frac{1}{\cos^2\alpha}(1-\frac{g}{4\pi})\right] \tag{1.2}$$

where the boson mass $\mathcal{M}$ is related to α as,

$$\mathcal{M} = 2m\cos\alpha$$

where α is between 0 and $\frac{\pi}{2}$. g is a coupling constant of the massive Thirring model with Johnson's normalization [4]. Here, one can easily check that there is only one bound state.

Since this eigenvalue equation is obtained with fermion antifermion Fock space only, one may say that this is a good approximate solution to the massive Thirring model. However, it turns out that the solution eq.(1.2) has all the proper behaviors of the weak and strong coupling limits. Instead, if one checks eq.(1.1) carefully, then one sees that the semiclassical result of eq.(1.1) does not have a proper weak coupling limit. There, the important point is that one has to take into account current regularizations in a correct way [3].

Further, Ogura, Tomachi and Fujita [5] estimated the effect of higher fermion antifermion Fock spaces (two fermion two antifermion Fock space) and proved that the interactions between two bosons are always repulsive. Therefore, it is confirmed that there is only one bound state in the massive Thirring model from the infinite momentum frame

prescription.

Here, a serious question arises. How about the Bethe ansatz solution for the massive Thirring model ? The Bethe ansatz wave function is well known to be exact. This is a strong reason why people have believed for almost two decades that the bound state spectrum obtained from the semiclassical approximation is exact in spite of the fact that they took into account only the lowest quantum fluctuations in the path integral.

In this paper, we reexamine the Bethe ansatz solutions for the massive Thirring model and discuss problems in the treatment by Bergknoff and Thacker [2]. In particular, we show that the *string* configurations taken by Bergknoff and Thacker do not satisfy the PBC equations. The reason why they have to introduce the *string* picture is because they solve the PBC equations for the density of states. Therefore, they could not determine proper rapidities for the positive energy particles.

It is now clear what one should do. One should solve the PBC equations directly for the rapidities (momenta) without referring to the density of states. This is what we have done in this paper. We have solved the PBC equations numerically. We consider a few hundred particles to a few thousand particles to make a vacuum. Then, we make one particle-one hole pairs, two particle two hole pairs and so on. It is found that there is only one bound state for one particle-one hole $(1p - 1h)$ configuration. There is no bound state for two particle two hole cases.

Further, the bound state energy calculated from the Bethe ansatz PBC equations turns out to be consistent with that of Fujita-Ogura's solution [eq. (1.2)] though we can solve only a limited region of the coupling constant.

Further, we find the boson boson scattering states in $2p-2h$ configurations. Here, it is important to note that the boson boson scattering states have rapidity variables which are all real, or at least if at all exist, a very small imaginary part. Therefore, there is no $string-$like solution which satisfies the PBC equations.

Therefore, we present some evidences that there is only one bound state in the massive Thirring model and that the semiclassical result by Dashen et al. is indeed an approximate solution.

In order that the paper can be understood in a better fashion, we comment that the interactions between particles in the massive Thirring model are always repulsive for the positive value of the coupling constant. This is trivial but quite important to understand the bound state problem of the massive Thirring model.

The paper is organized as follows. In the next section, we briefly explain the Bethe ansatz solution of the massive Thirring model. Then, section 3 treats the periodic boundary condition of the solution. Also, the regularization in this model is discussed. In section 4, numerical results of the bound state spectrum are presented. In particular, we treat

the mass of the boson which is the only bound state in this model. In section 5, we discuss the boson boson scattering states in $2p - 2h$ configurations. Section 6 summarises what we have understood and clarified from this work.

2. Massive Thirring model and Bethe ansatz solutions

The massive Thirring model is a 1+1 dimensional field theory with current current interactions [6]. Its lagrangian density can be written as

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m_0)\psi - \frac{1}{2} g_0 j^\mu j_\mu \tag{2.1}$$

where the fermion current j_μ is written as

$$j_\mu =: \bar{\psi}\gamma_\mu \psi : . \tag{2.2}$$

Choosing a basis where γ_5 is diagonal, the hamiltonian is written

$$H = \int dx \left[-i(\psi_1^\dagger \frac{\partial}{\partial x}\psi_1 - \psi_2^\dagger \frac{\partial}{\partial x}\psi_2) + m_0(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g_0 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right] . \tag{2.3}$$

Now, we define the number operator N as

$$N = \int dx \psi^\dagger \psi . \tag{2.4}$$

This number operator N commutes with H. Therefore, when we construct physical states, we must always consider physical quantities with the same particle number N as the vacuum. For different particle number state, the vacuum is different and thus the model space itself is different.

The hamiltonian eq.(2.3) can be diagonalized by the Bethe ansatz wave functions. Here, we do not repeat the way to construct the Bethe ansatz wave functions since it is very well written in Thacker's review paper

[7]. Therefore, for detail discussions, the reader should refer to his paper.

The Bethe ansatz wave function $\Psi(x_1, ..., x_N)$ for N particles can be written as

$$\Psi(x_1, ..., x_N) = \exp(im_0 \sum x_i \sinh \beta_i) \prod_{1 \le i < j \le N} [1 + i\lambda(\beta_i, \beta_j)\epsilon(x_i - x_j)] \tag{2.5}$$

where β_i is related to the momentum k_i and the energy E_i of i-th particle as

$$k_i = m_0 \sinh \beta_i. \tag{2.6a}$$

$$E_i = m_0 \cosh \beta_i. \tag{2.6b}$$

where β_i's are complex variables.

$\epsilon(x)$ is a step function and is defined as

$$\epsilon(x) = \begin{cases} -1 & x < 0 \\ \\ 1 & x > 0. \end{cases} \tag{2.7}$$

$\lambda(\beta_i, \beta_j)$ is related to the phase shift function $\phi(\beta_i - \beta_j)$ as

$$\frac{1 + i\lambda(\beta_i, \beta_j)}{1 - i\lambda(\beta_i, \beta_j)} = \phi(\beta_i - \beta_j). \tag{2.8}$$

The phase shift function $\phi(\beta_i - \beta_j)$ can be explicitly written as

$$\phi(\beta_i - \beta_j) = -2 \tan^{-1} \left[\frac{1}{2} g_0 \tanh \frac{1}{2} (\beta_i - \beta_j) \right]. \tag{2.9}$$

In this case, the eigenvalue equation becomes

$$H \mid \beta_1 ... \beta_N >= (\sum_{i=1}^{N} m_0 \cosh \beta_i) \mid \beta_1 ... \beta_N > \tag{2.10}$$

where $\mid \beta_1 ... \beta_N >$ is related to $\Psi(x_1, ..., x_N)$ as

$$\mid \beta_1 ... \beta_N >= \int dx_1 ... dx_N \Psi(x_1, ..., x_N) \prod_{i=1}^{N} \psi^\dagger(x_i, \beta_i) \mid 0 > . \qquad (2.11)$$

Also, $\psi(x, \beta)$ can be written in terms of $\psi_1(x)$ and $\psi_2(x)$ as,

$$\psi(x, \beta) = e^{\frac{\beta}{2}} \psi_1(x) + e^{-\frac{\beta}{2}} \psi_2(x). \qquad (2.12)$$

From the definition of the rapidity variable β_i's, one sees that for positive energy particles, β_i's are real while for negative energy particles, β_i takes the form $i\pi - \alpha_i$ where α_i's are real. Therefore, in what follows, we denote the positive energy particle rapidity by β_i and the negative energy particle rapidity by α_i.

3. Periodic Boundary Conditions and Regularizations

The Bethe ansatz wave functions satisfy the eigenvalue equation [eq.(2.10)]. However, they still do not have proper boundary conditions. The simplest way to define field theoretical models is to put the theory in a box of length L and impose periodic boundary conditions (PBC) on the states.

Therefore, we demand that $\Psi(x_1, .., x_N)$ be periodic in each argument x_i. This gives the boundary condition

$$\Psi(x_i = 0) = \Psi(x_i = L). \tag{3.1}$$

This leads to the following PBC equations,

$$\exp(im_0 L \sinh\beta_i) = \exp(-i\sum_j \phi(\beta_i - \beta_j)). \tag{3.2}$$

Taking the logarithm of eq.(3.2), we obtain

$$m_0 L \sinh\beta_i = 2\pi n_i - \sum_j \phi(\beta_i - \beta_j). \tag{3.3}$$

where n_i's are integer. These are equations which we should now solve.

Before going to construct physical states, we discuss the regularization of the fermion current. This is somehow a complication of the massive Thirring model which people often overlook. As Klaiber explained in his paper [8], the Thirring model has an ambiguity that comes from current regularizations. For any field theories with local gauge invariance, there is no ambiguity concerning the current regularization since

one has to make gauge invariant regularizations. If one makes gauge invariant regularizations, then one obtains physical quantities which do not depend on the choice of the regularization methods. The Thirring model has no local gauge invariance and thus may well have some ambiguity that arises from the way one makes regularizations.

In the treatment of the Bethe ansatz wave functions, we have made a regularization when constructing $\Psi(x_1, .., x_N)$. There, we assume the following identity for the step function $\epsilon(x)$,

$$\frac{d}{dx}\epsilon(x) = 2\delta(x), \qquad \delta(x)\epsilon(x) = 0. \tag{3.4}$$

This regularization involves only space coordinate. This indicates that the regularization employed here must be Schwinger's regularization [9]. For Schwinger's regularization, we denote the coupling constant by g_0. In this case, the value of g_0 varies from $-\frac{1}{2}$ to ∞. On the other hand, there are other regularizations. In particular, Johnson's regularization is most popular [4]. There, the current is regularized with the point splitting of space and time in a symmetric fashion. This has some advantage in that the current conservation is preserved. The coupling constants in the two different regularizations (Schwinger and Johnson) are related to each other as follows,

$$g_0 = \frac{2g}{2-\frac{g}{\pi}} \tag{3.5}$$

where g denotes the coupling constant with Johnson's regularization.

For Johnson's regularization, as Klaiber states, free fermion basis states lead to Johnson's regularization. Therefore, if one works in perturbation theory, then one uses automatically Johnson's regularization [10].

In the Introduction, we mentioned that the semiclassical result of eq.(1.1) does not have the right weak coupling limit. Dashen et al. expanded eq.(1.1) in terms of g_0 for the weak coupling limit and obtained the result as

$$\mathcal{M} = m\left(2 - g_0^2 + \frac{4}{\pi} g_0^3 + ..\right). \tag{3.6}$$

On the other hand, the perturbative calculation of the bound state spectrum is found

$$\mathcal{M} = m\left(2 - g^2 + \frac{4}{\pi} g^3 + ..\right). \tag{3.7}$$

The important point is that the perturbative treatment automatically employs Johnson's regularization and thus the result should be written by the coupling constant g in eq.(3.7). Thererfore, it is clear that eq.(3.6) does not agree with eq.(3.7) and thus eq.(1.1) does not have the right behavior of the weak coupling limit at the order of $O(g^3)$. The detailed discussions can be found in ref.[5].

Throughout this paper, we use Schwinger's normalization g_0. Later in this paper, we often specify the table and the figures by the coupling constant $\frac{g}{\pi}$ (Johnson's normalization). This is because it is easier to compare the numerical results with Fujita-Ogura's solutions. But the

value of $\frac{g}{\pi}$ is easily converted into g_0 by eq. (3.5).

4. Numerical Solutions

Now, we are ready to construct physical states. The parameters we have here are the box length L and the particle number N. In this case, the density of the system ρ becomes

$$\rho = \frac{N}{L}. \tag{4.1a}$$

Here, the system is fully characterized by the density ρ. For later convenience, we define the effective density ρ_0 as

$$\rho_0 = \frac{N_0}{L_0} \tag{4.1b}$$

where L_0 and N_0 are defined as $L_0 = m_0 L$ and $N_0 = \frac{1}{2}(N-1)$, respectively.

(a) Vacuum state

First, we want to make a vacuum. We write the PBC equations for the vacuum which is filled with negative energy particles (

$\beta_i = i\pi - \alpha_i$),

$$\sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left[\frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right], \qquad (i = 1, .., N). \tag{4.2}$$

Now, n_i runs as

$$n_i = 0, \pm 1, \pm 2, ..., \pm N_0.$$

Therefore, n_i can be replaced by i and thus eq.(4.2) becomes

$$\sinh \alpha_i = \frac{2\pi i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left[\frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right], \qquad (i = 0, \pm 1, .., \pm N_0). \tag{4.3}$$

We fix the values of L_0 and N, and then can solve eq.(4.3). This determines the vacuum. In this case, the vacuum energy E_v can be written as

$$E_v = - \sum_{i=-N_0}^{N_0} m_0 \cosh \alpha_i. \tag{4.4}$$

To describe physical states, we have to renormalize the energy to some physical point. Therefore, m_0 itself does not play any important role.

In fig.1, we show how the vacuum and other particle-hole states are constructed. Fig.1a shows the vacuum state. Depending on the value of the coupling constant, the shape changes. This has no ambiguity and one can also make the vacuum with solving for the density of states.

(b) $1p-1h$ configuration

Next, we want to make one particle-one hole ($1p-1h$) state. That is, we take out one negative energy particle (i_0-th particle) and put it into a positive energy state. In this case, the PBC equations become

$i \neq i_0$

$$\begin{aligned}\sinh\alpha_i &= \frac{2\pi i}{L_0} - \frac{2}{L_0}\tan^{-1}\left[\frac{1}{2}g_0\coth\frac{1}{2}(\alpha_i+\beta_{i_0})\right] \\ &-\frac{2}{L_0}\sum_{j\neq i,i_0}\tan^{-1}\left[\frac{1}{2}g_0\tanh\frac{1}{2}(\alpha_i-\alpha_j)\right] \quad (4.5\mathrm{a})\end{aligned}$$

$i = i_0$

$$\sinh\beta_{i_0} = \frac{2\pi i_0}{L_0} + \frac{2}{L_0}\sum_{j\neq i_0}\tan^{-1}\left[\frac{1}{2}g_0\coth\frac{1}{2}(\beta_{i_0}+\alpha_j)\right] \quad (4.5\mathrm{b})$$

where β_{i_0} can be a complex variable as long as it can satisfy eqs.(4.5).

These PBC equations determine the energy of the one particle-one hole states which we denote by $E_{1p1h}^{(i_0)}$,

$$E_{1p1h}^{(i_0)} = m_0\cosh\beta_{i_0} - \sum_{\substack{i=-N_0\\ i\neq i_0}}^{N_0} m_0\cosh\alpha_i. \qquad (4.6)$$

It is important to notice that the momentum allowed for the positive energy state must be determined by the PBC equations. Also, the momenta occupied by the negative energy particles are different from the vacuum case.

The lowest configuration one can consider is the case in which one takes out $i = 0$ particle and puts it into the positive energy state. This is shown in fig.1b. This must be the first excited state since it has a symmetry of $\alpha_i = -\alpha_{-i}$. We call this state "symmetric" since it has a left-right symmmetry in fig.1b.

Next, we consider the following configurations in which we take out $i = \pm 1, \pm 2, ..$ particles and put them into the positive energy state. This is shown in fig.1c. These are configurations we can build up for one particle-one hole state. We should note here that we cannot make one particle state or one hole state since there the particle number is different from the vacuum. Since the hamiltonian of the massive Thirring model commutes with particle number operator, we should always stay in the same particle number as the vacuum. In this sense, we have lost a simplest renormalization point. It would have been ideal if we could make one particle scattering (or continuum) state to which we renormalize the physical mass. In this case, we could have had a more predictive power to describe the mass of the particle hole state.

Therefore, we should find out another way to renormalize our calculated energy. Fortunately, we find that the continuum states of the one particle-one hole appear very clearly. Therefore, we can renormalize the physical mass to this point as we will discuss it

later.

(c) $2p - 2h$ configurations

In the same way as above, we can make two particle-two hole $(2p-2h)$ states. Here, we take out i_1-th and i_2-th particles and put them into positive energy states. The PBC equations for the two particle-two hole states become

$$i \neq i_1, i_2$$

$$\begin{aligned}
\sinh \alpha_i &= \frac{2\pi i}{L_0} - \frac{2}{L_0} \tan^{-1}\left[\frac{1}{2} g_0 \coth \frac{1}{2}(\alpha_i + \beta_{i_1})\right] \\
&- \frac{2}{L_0} \tan^{-1}\left[\frac{1}{2} g_0 \coth \frac{1}{2}(\alpha_i + \beta_{i_2})\right] \\
&- \frac{2}{L_0} \sum_{j \neq i, i_1, i_2} \tan^{-1}\left[\frac{1}{2} g_0 \tanh \frac{1}{2}(\alpha_i - \alpha_j)\right]
\end{aligned} \tag{4.7a}$$

$$i = i_1$$

$$\begin{aligned}
\sinh \beta_{i_1} &= \frac{2\pi i_1}{L_0} + \frac{2}{L_0} \tan^{-1}\left[\frac{1}{2} g_0 \tanh \frac{1}{2}(\beta_{i_1} - \beta_{i_2})\right] \\
&+ \frac{2}{L_0} \sum_{j \neq i_1, i_2} \tan^{-1}\left[\frac{1}{2} g_0 \coth \frac{1}{2}(\beta_{i_1} + \alpha_j)\right]
\end{aligned} \tag{4.7b}$$

$$i = i_2$$

$$\begin{aligned}
\sinh \beta_{i_2} &= \frac{2\pi i_2}{L_0} + \frac{2}{L_0} \tan^{-1}\left[\frac{1}{2} g_0 \tanh \frac{1}{2}(\beta_{i_2} - \beta_{i_1})\right] \\
&+ \frac{2}{L_0} \sum_{j \neq i_1, i_2} \tan^{-1}\left[\frac{1}{2} g_0 \coth \frac{1}{2}(\beta_{i_2} + \alpha_j)\right]
\end{aligned} \tag{4.7c}$$

In this case, the energy of the $2p-2h$ states $E^{(i_1,i_2)}_{2p2h}$ becomes

$$E^{(i_1,i_2)}_{2p2h} = m_0 \cosh \beta_{i_1} + m_0 \cosh \beta_{i_2} - \sum_{\substack{i=-N_0 \\ i \neq i_1, i_2}}^{N_0} m_0 \cosh \alpha_i. \tag{4.8}$$

Here, we note that the symmetric case ($i_1 = -i_2$) gains the energy and therefore is lower than other asymmetric cases of $2p-2h$ states.

Higher particle-hole states are constructed just in the same way as above. But it turns out that already two particle two hole states do not give any bound states. Therefore, it is not worthwhile carrying out numerical calculations of higher particle hole states.

Now, we discuss numerical results of the particle-hole energy. We solve the PBC equations by iterations. Since the equations are nonlinear, it is nontrivial to solve them by iterations. Indeed, simple-minded iteration procedures do not give good convergent solutions.

(d) A new iteration method

Here, we briefly explain how we solve the nonlinear coupled equations by computor. The type of equation we want to solve can be schematically written as

$$\boldsymbol{f} = G(\boldsymbol{f}) \tag{4.9}$$

where $\boldsymbol{f} = (f_1, f_2, .., f_N)$ are the N variables that should be determined. G is some function. Now, we want to solve it by iterations. The simple-minded iteration equation we can make is

$$\boldsymbol{f}^{(n+1)} = G(\boldsymbol{f}^{(n)}) \tag{4.10}$$

where we start from some initial value of $\boldsymbol{f}^{(0)}$. However, this seldom gives a convergent result. Here, instead of eq.(4.10), we use the following equation,

$$\boldsymbol{f}^{(n+1)} = G\left(s\boldsymbol{f}^{(n)} + (1-s)\boldsymbol{f}^{(n-1)}\right) \tag{4.11}$$

where s is a free parameter that should be chosen so that it gives a convergent result. Indeed, if we vary the value of s around $s \sim 0.1$, then we get good convergent results. In particular, for the asymmetric $1p - 1h$ states, the introduction of s is essential.

Before going to discussions of our calculated spectrum, we should note that we use mostly $\frac{g}{\pi}$ instead of g_0 in the tables and figures of the calculations just for convenience as mentioned before. Also, in what follows, we treat the cases in which there is some possibility for many bound states. This corresponds to rather strong coupling regions. Therefore, we only focus on the cases with the coupling constant $\frac{g}{\pi}$ which is larger than 0.8. This is because, in this region, the semiclassical calculation predicts many bound states. Also, there, twice of the first excited state energy (boson mass) is lower

than a half of the free fermion antifermion mass and thus there is a chance of having some bound states of bosons.

(e) Energy spectrum

In table 1, we show the calculated energies (raw data) of E_v, $E^{(0)}_{1p1h}$, $E^{(n)}_{1p1h}$ and $E^{(n_1,n_2)}_{2p2h}$ for several values of the coupling constants with the particle number $N = 1601$. Here, we can put $m_0 = 1$ without loss of generality.

For $E^{(n)}_{1p1h}$ and $E^{(n_1,n_2)}_{2p2h}$, we show the lowest six states just to see the structure of the spectrum. As can be seen from this table, there is a finite jump between E_v, $E^{(0)}_{1p1h}$, $E^{(1)}_{1p1h}$. However, the differences among $E^{(n)}_{1p1h} \quad (n = 1, 6)$ are always some small number which is just $\frac{2\pi}{L_0}$, corresponding to the smallest momentum in this calculation. The same phenomena occur to the $2p - 2h$ cases. These states correspond to the continuum states. This situation is better seen if we plot them in the figure.

In fig.2, we show the spectrum of the $1p - 1h$ and $2p - 2h$ states at the coupling constant $\frac{g}{\pi} = 1.25$ as an example (particle number N is 1601, and L_0 is 110). One clearly sees that there is one bound state and then continuum (or scattering) states start. The first bunch of continuum states correspond to the $1p - 1h$ states. Therefore, the lowest part of the continuum state can be identi-

fied as the free fermion and free antifermion state at rest, which should be just $\mathcal{M} = 2m$ where m denotes a physical fermion mass. Namely, we now know that we find a renormalization point of the physical mass. The $1p - 1h$ continuum energy should start from the free fermion antifermion mass, that is twice the fermion mass. Therefore, the physical fermion mass m can be written as

$$m = \frac{1}{2}\left(E^{(1)}_{1p1h} - E_v\right).$$

Then, another bunch of continuum states appear which correspond to the $2p - 2h$ states. It is amusing to notice that the energy of the $2p - 2h$ continuum is just twice of the $1p - 1h$ state energy. This is quite important since this indicates that we solve the PBC equations properly. Also, the physical fermion mass which we identify by the continuum state of $1p - 1h$ configuration is indeed justified. From the figure, it is clear that the $2p - 2h$ configurations do not have any bound states but they are two fermion two antifermion free states. Therefore, we can calculate the bound state mass of $1p - 1h$ state with respect to the free fermion antifermion masses.

Further, there is another bunch of continuum states which correspond to the boson boson scattering states though these states are not shown in fig.2. These boson boson scattering states appear near the twice of the boson mass. As will be discussed later, the

boson boson scattering states behave quite differently from the two fermion or four fermion scattering states.

(f) Bound state mass in $1p-1h$ states

In fig.3, we show the calculated results of excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ for $\frac{g}{\pi} = 0.8$ ($g_0 = 4.19$), $\frac{g}{\pi} = 1$ ($g_0 = 6.28$) and $\frac{g}{\pi} = 1.25$ ($g_0 = 10.5$) cases as the function of $\rho_0 = \frac{N_0}{m_0 L}$. Here, $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ are defined as the $1p-1h$ state energies with respect to the vacuum,

$$\Delta E^{(0)}_{1p1h} = E^{(0)}_{1p1h} - E_v \tag{4.12a}$$

$$\Delta E^{(1)}_{1p1h} = E^{(1)}_{1p1h} - E_v. \tag{4.12b}$$

As can be seen from fig.3, the excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ have almost the same slope between them if we plot them in log-log scale as the function of ρ_0. This suggests that we can write them as

$$\Delta E^{(0)}_{1p1h} = A_0 + B_0 \rho_0^{\alpha} \tag{4.13a}$$

$$\Delta E^{(1)}_{1p1h} = A_1 + B_1 \rho_0^{\alpha} \tag{4.13b}$$

where $A_i,\ B_i\ \ (i = 0, 1)$ are constant but depend on the coupling constant g_0. The important point is that $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ have the same slope α. Here, the values of the α for $\frac{g}{\pi} = 0.8$, 1.0 and

1.25 turn out to be

$$\alpha = 0.42 \qquad \text{for} \quad \frac{g}{\pi} = 0.8 \tag{4.14a}$$

$$\alpha = 0.45 \qquad \text{for} \quad \frac{g}{\pi} = 1.0 \tag{4.14b}$$

$$\alpha = 0.50 \qquad \text{for} \quad \frac{g}{\pi} = 1.25. \tag{4.14c}$$

In fig.4, we show the same excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ as the function of ρ_0^α for $\frac{g}{\pi} = 0.8$, 1.0 and 1.25 cases. From this figure, one sees that calculated points are almost on the straight line. Also, one notices that the calculated results are consistent with $A_i = 0$.

We note here that α depends on the coupling constant g_0 and becomes unity when $g_0 \to \infty$. But it is always smaller than unity. In this case, we can take the field theory limit $\rho \to \infty$. This can be seen in the following way. Let us take the $\Delta E^{(0)}_{1p1h}$ case, for example. If we write the bare mass m_0 explicitly, then $\Delta E^{(0)}_{1p1h}$ can be written as

$$\Delta E^{(0)}_{1p1h} = m_0 \left(A_0 + B_0 \left(\frac{\rho}{m_0} \right)^\alpha \right). \tag{4.15}$$

Now, we want to let $\rho \to \infty$, keeping $\Delta E^{(0)}_{1p1h}$ finite. Since α is smaller than unity, we can make a fine-tuning of m_0 such that

$$m_0^{1-\alpha} \rho^\alpha = \text{finite}.$$

This means that we should let $m_0 \to 0$, and thus the second term of eqs. (4.13) becomes dominant in the field theory limit. In this case, we can identify the mass of the bound state $\mathcal{M}$ as

$$\mathcal{M} = 2m \lim_{\rho\to\infty} \left(\frac{\Delta E^{(0)}_{1p1h}}{\Delta E^{(1)}_{1p1h}} \right) = 2m \frac{B_0}{B_1}. \tag{4.16}$$

For $\frac{g}{\pi} = 0.8$, 1.0 and 1.25 cases, we find

$$B_0 = 2.55, \qquad B_1 = 5.06 \qquad \text{for} \quad \frac{g}{\pi} = 0.8 \tag{4.17a}$$

$$B_0 = 2.25, \qquad B_1 = 6.0 \qquad \text{for} \quad \frac{g}{\pi} = 1.0 \tag{4.17b}$$

$$B_0 = 1.63, \qquad B_1 = 7.0 \qquad \text{for} \quad \frac{g}{\pi} = 1.25. \tag{4.17c}$$

Thus, the mass of the bound states for $\frac{g}{\pi} = 0.8$, 1.0 and 1.25 becomes

$$\mathcal{M} = 1.01 \qquad \text{for} \quad \frac{g}{\pi} = 0.8 \tag{4.18a}$$

$$\mathcal{M} = 0.75 \qquad \text{for} \quad \frac{g}{\pi} = 1.0 \tag{4.18b}$$

$$\mathcal{M} = 0.47 \qquad \text{for} \quad \frac{g}{\pi} = 1.25 \tag{4.18c}$$

Here, we comment on the minimum theoretical errors which may arise from the minimum momentum of the calculation $\frac{2\pi}{L_0}$. This gives rise to an error for the mass

$$\Delta\mathcal{M} \approx \frac{2\pi}{L_0} \frac{2}{\Delta E^{(1)}_{1p1h}} 2m \approx 0.08m.$$

However, the above errors may not be the largest one. The larger errors may well come from the fact that we have still not yet

reached sufficiently large values of N and ρ_0 such that $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ have the same slope α.

The calculated boson mass (eqs.(4.18)) should be compared to those predicted by other methods. In table 2, we show the predictions by the infinite momentum frame calculation (Fujita and Ogura), the semiclassical method (Dashen et al.) and the Bethe ansatz solution with *string* hypothesis (Bergknoff and Thacker). The comparison can be better seen if we plot them in the figure. In fig.5, we show the boson mass predicted by different methods for the whole range of the coupling constant. The black circles are the present calculation. The fig.5 indicates that the present calculation is consistent with that predicted by Fujita and Ogura (the solid line). Also, one can say that the difference between the present result and the semiclassical one (the dashed line) is not very large and is probably within an error of our calculation. However, the predicted value of Bergknoff and Thacker (the dashed-dotted line) is very different from the present result. This is quite important since we solve the same PBC equations as Bergknoff and Thacker, keeping the same coupling constant g_0. The only difference between the present calculation and Bergknoff and Thacker must lie in the treatment of determining the rapidity of the positive energy particles. There, they employed the

string hypothesis since their method that uses the density of states cannot determine the rapidity of positive energy particle in the particle-hole excitations. This indicates that the *string* hypothesis may not be a very good approximate scheme for the massive Thirring model even though it is a good working tool for the nonlinear Schrödinger model. In ref.[15], it is proved that, in the strong coupling limit, the *string* picture does not satisfy the PBC equation.

Note that Bergknoff and Thacker claim that they reproduced the semiclassical results by Dashen et al.. There, when they compare their spectrum with the semiclassical result, they have made a renormalization of the coupling constant g. However, this is not justified since Dashen et al. use Schwinger's normalization of the coupling constant, namely, the same g_0 as Bergknoff and Thacker. Here, it is clear since we used Bergknoff-Thacker's formula, keeping the same coupling constant g_0 as appeared in this paper since we solved the same equation with the same boundary conditions.

At this point, we comment on the renormalization of the coupling constant. Here, we do not have to make any renormalizations of the coupling constant when calculating physical quantities. Only the mass renormalization is needed. The only important point is that one has to know which regularization of the current one has

employed in his calculation.

Now, we discuss the limit of our calculation. Unfortunately, we cannot find solutions beyond some value of ρ_0. That is entirely due to the problem of our computer program. Until now, we do not find any better way of calculating the cases with higher values of ρ_0. But as far as the $\frac{g}{\pi} = 0.8$, 1.0 and 1.25 cases are concerned, we have reached relatively large value of ρ_0 even though it is not yet sufficient. This can be seen in the following way. In order that the vacuum can be constructed in a proper fashion, the following conditions must be satisfied,

$$\frac{2\pi}{L} \ll m_0 \ll \frac{2\pi N_0}{L}. \tag{4.19}$$

For the cases of $\frac{g}{\pi} = 0.8$, 1.0 and 1.25, we have

$$\begin{aligned} L_0 &= m_0 L \approx 10 \sim 100 \\ N_0 &= 800. \end{aligned}$$

Therefore, the above equation becomes

$$(0.6 \sim 0.06) \ll 1 \ll (50 \sim 500). \tag{4.20}$$

Here, the left-hand inequality is not very well satisfied. This corresponds to the large value of ρ_0. This may well generate larger errors than $\Delta\mathcal{M}$. To overcome this difficulty, we have to increase the number of particle N by order of magnitude. From these considerations, we can say that our calculated value of the bound

state mass is reliable only to some extent to which the slopes of $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ are the same. As mentioned above, the two slopes are still slightly different. In particular, the two slopes for $\frac{g}{\pi} = 1.25$ case are still appreciably different from each other.

In fig.6, we show the calculated results for $\frac{g}{\pi} = 1.5, \ 1.7$. As can be seen from this figure, we have not yet reached sufficiently large values of ρ_0. Therefore, the slopes of the $\Delta E^{(0)}_{1p1h}$ and the $\Delta E^{(1)}_{1p1h}$ are still quite different from each other. Thus, we cannot make the field theory limit for these cases. This is entirely due to the poorness of our computer program. Indeed, for the coupling constants $\frac{g}{\pi}$ larger than 1.8, the situation is even worse. We can not find any reasonable solutions yet. In this respect, the calculations presented here are only very limited. This is certainly connected to the fact that the PBC equations are highly nonlinear.

5. Excitation energies in $2p-2h$ states

As shown in fig.2, the excitation energies of the $2p-2h$ states always appear at the energy of four times physical fermion mass. This is one of the strong evidences that we solve the PBC equations properly.

However, the above calculations do not include the important states of $2p - 2h$ configurations, that is, solutions for boson-boson scattering states near the energy of the twice of the boson mass.

In this section, we present the boson boson scattering states which are found in the $2p - 2h$ configurations.

First, we note that the iteration procedure described in section 4(d) with real rapiditiy variables always give the four fermion states as we saw before.

Here, we make the rapidity variables all complex so that we can find the $string-$like solutions which should correspond to the boson boson scattering states. Now, it turns out that, in order to find the boson boson states, we must vary the initial values of the rapidities in the iteration procedure. Depending on the initial values, we find the boson boson scattering states in the one case, but find the four fermion states in the other case.

In any case, once we know how to find the boson boson states, then, we can obtain the boson boson state energy for any states we need. Here, we note that all the resulting rapidity variables are found to be real, even though we start from the complex initial values of the rapidity.

Therefore, this shows that there is no $string-$like solution which satisfies the PBC equation. The imaginary part of the rapidity, if at all exist, should be quite small.

In Table 3, we show the first six excitation energies of the boson boson states E_{2p2h}^{BB} for several values of the coupling constants. This clearly shows that these states are scattering states (unbound states) and make a continuum spectrum. It is quite interesting to observe that these continuum states differ from each other by the energy unit which is smaller than the case with four fermion states. In fact, the energy unit of the continuum state for the four fermion states is $\frac{4\pi}{L_0}$ (see Table 1) while the energy unit for the boson boson states is a factor of 5 to 10 smaller than $\frac{4\pi}{L_0}$. This suggests that these states are indeed quite different from four fermion states.

Now, it should be fair to note that it is indeed quite difficult to find the boson boson states by computor. This is mainly because we do not know *a priori* which of the states $(n, -n)$ is the lowest. The lowest energy of the boson boson scattering state can be obtained only after we obtain all the boson boson state energies. Therefore, it is always very much cumbersome to find the lowest energy of the boson boson scattering states. This must be connected to the fact that the boson boson scattering states are constructed by the two step processes, first by making one boson state and then by making the boson boson scattering states.

In Table 4, we summarize the calculated results of the excitation energies for several values of the coupling constants as well as for the several

cases of the particle numbers N and the box sizes L_0. Here, we plot the lowest state of the boson boson excitation energy ΔE_{2p2h}^{BB} with respect to the vacuum, and the first excited state (boson state) $\Delta E_{1p1h}^{(0)}$ as well so that we can compare the calculated results of the boson boson state energy with twice of the boson mass. Also, for comparison, we show the lowest state of the two fermion scattering state $\Delta E_{1p1h}^{(1)}$, and the lowest state of the four fermion scattering state $\Delta E_{2p2h}^{(1,-1)}$. As can be seen from the Table 4, the boson boson states appear lower than the twice of the boson mass. In particular, depending on the density ρ_0, the boson boson state energy becomes much lower than the twice of the boson mass. Also, the dependence of the boson boson state energy on the ρ_0 is in general quite different from the $1p-1h$ state energies. We do not fully understand the reason why there is such a significant difference between the boson boson state energy and the free fermion ($1p-1h$ and $2p-2h$ states) energies.

From the present calculations, we get to know that the boson boson state energy depends strongly on the box size L_0. Unfortunately, in the present calculation, the box size is much too small. In particular, for large ρ_0, the value of L_0 is too small to satisfy the criteria of eq.(4.19). This may be one of the reasons why we obtain the very low excitation energy of the boson boson states.

Since the dependence of the boson boson excitation energy $\Delta E_{2p2h}^{(BB)}$ on

the density ρ_0 is quite different from those of $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$, we cannot, therefore, make the field theory limit as we have made it for $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ cases. As stated above, the values of N and L which we took here are still too small in order to obtain some quantitative numbers for $\Delta E^{(BB)}_{2p2h}$. In this respect, we only get some qualitative pictures of the boson boson scattering states.

Finally, we want to make a comment on the validity of the boson boson state energy. In ref.[15], the strong coupling expansion is presented. There, the excitation energy is evaluated analytically. This analytic expression is compared to our calculations of the boson boson state energy. It turns out that the both calculations agree with each other quite well for the large values of L_0.

6. Conclusions and discussions

We have presented a new interpretation of the Bethe ansatz solutions of the massive Thirring model. Here, we solve the PBC equations directly but numerically without referring to the density of states or *string* hypothesis. It is found that the Bethe ansatz solutions produce one bound state (a boson). This spectrum as the function of the coupling constant is consistent with Fujita-Ogura's solution.

Also, it is shown that the *string* configurations taken by Bergknoff and Thacker do not satisfy the PBC equations and thus their *string* is not a solution of the PBC equations. In this way, the present result rules out a belief that the semiclassical result for the massive Thirring model is exact.

Here, we want to give an intuitive argument why there is only one bound state from the PBC equations. The vacuum is represented by fig.1a. This figure shows that the left-right symmetry ($\alpha_i = -\alpha_{-i}$) is preserved there. Now, we make one particle one hole state as shown in fig.1b and 1c. The fig.1b has the left-right symmetry and therefore it gains the energy. The important point is that there is only one state that has this symmetry. In fact, the fig.1c does not have the left-right symmetry and therefore its energy is rather high. This obviously gives a continuum spectrum. Now, let us consider the two particle-two hole states. In fig.1d, we show only the symmetric case since the asymmetric

configurations do not gain the energy. Here, the important point is that there is no special configuration which differs from others. The lowest configuration is different from the next lowest only by $\frac{4\pi}{L_0}$ and so on. Therefore, it is clear that these $2p-2h$ states should describe some continuum states. For higher particle-hole states, the situation is the same as the $2p-2h$ states. Therefore, it is qualitatively clear why there is only one bound state from the PBC equations. This is indeed confirmed by numerical calculations.

Also, the strong coupling expansion is performed in ref.[15] and the analytic expressions are obtained for the vaccum state energy as well as the boson boson scattering states. There, it turns out that the boson boson scattering states which are made of continuum states coincide with the twice of the boson mass. Therefore, we also learn from the strong coupling expansion that the $2p-2h$ states do not give any bound states, to say the least, there is no bound state found in the analytic expressions of the strong coupling expansion.

This should naturally bring up many problems and questions concerning those methods or solutions which agree with the semiclassical results [11,12,13,14,17].

First, we comment on Baxter's solution to the Heisenberg XYZ model [11]. This model is shown to be equivalent to the massive Thirring model [12]. Baxter's solution is indeed exact. However, he presented his

solution only for the largest eigenvalue of the XYZ hamiltonian. After Baxter's solution, Johnson, Krinsky and McCoy [13] obtained the next largest eigenvalue by emloying the *string* hypothesis. Again, the same approximate scheme as the *string* is used there. As we showed in this paper, these states constructed from the *string* hypothesis correspond to boson-boson scattering states. In quantum mechanical terminology, they may correspond to real quantum states, but they are not bound states! Therefore, the solution of the Heisenberg XYZ model has the same problem as Bergknoff-Thacker's solution.

Now, we want to discuss the S-matrix method by Zamolodchikov and Zamolodchikov [14]. This factorized S-matrix method is also known to give the same spectrum as the semiclassical result for the sine-Gordon field theory or the massive Thirring model. Concerning the factorization of the S-matrix for the particle-particle scattering in the massive Thirring model, one may convince oneself that the factorization is indeed satisfied.

However, there is a serious problem for the S-matrix factorization of the particle hole scattering. The problem is that the rapidity variables determined for $n-$particle $n-$hole states are different from each other as well as those determined for the vacuum. Since the Lagrangian of the massive Thirring model satisfies the charge conjugation, one tends to believe that the crossing symmetry should be automatically

satisfied. Indeed, the crossing symmetry itself should hold. However, we should be careful whether the crossing symmetry can commute with the factorization of the S-matrix or not. Recent calculations in ref.[16] show that the crossing symmetry and the factorization of the S-matrix do not commute with eath other. Therefore, it turns out that the S-matrix factorization for the particle hole scattering does not hold. In a sense, it is reasonable that the S-matrix factorization is consistent with the semiclassical results since it is indeed due to the consequence of the neglect of the operartor commutability.

Also, we discuss the results which are obtained from the inverse scattering methods [17]. This also gives the same spectrum as the semiclassical result. In this case, one can start from the soliton and antisoliton scattering. Therefore, there is no problem concerning the crossing symmetry. However, in this case, one does not know how to quantize the solitons and antisolitons in a proper fashion. Therefore, it is natural that the inverse scattering treatment remains semiclassical for the massive Thirring model and thus the result of the inverse scattering result agrees with the spectrum of the WKB method.

At this point, it should be fair to comment on the physical implication of the present result. Though the present paper shows that the excitation energy spectrum of sine-Gordon/massive Thirring model obtained by the WKB method is not exact, the physical significance of this proof

is not very great. We show that the boson boson states are all scattering states (unbound) while the WKB method insists that the lowest one of the boson boson states is a bound state. This energy difference is very tiny, and further any physical effects of this difference may be of a minor importance. But we simply stress that the WKB result is not exact but gives an approximate solution to the model.

There are, however, still many things to be done. In particular, it is very interesting to find analytic solutions of the PBC equations. Up to now, we do not know them. The vacuum itself is solved analytically with the help of the density of states and the vacuum energy is obtained analytically. However, it seems still quite difficult to solve analytically the PBC equations for particle-hole configurations, though we believe that the analytic solution must exist and should be found before long.

Finally, we make a comment as to whether there exist any exact solutions of the bound state spectrum in the massive Thirring model. Since the semiclassical result turns out to be not exact, it should be interesting to check whether Fujita-Ogura solution is exact or not. Although this spectrum of eq.(1.2) has all the nice featrures of the weak as well as strong coupling regions, there is no proof that the solution is exact. Therefore, it should be challenging to prove or disprove the exactness of the spectrum of eq.(1.2) since this is the only candidate that is still left undecided.

Acknowledgments: We thank C. Itoi, F. Lenz, H. Mukaida, A. Ogura and R. Woloshyn for stimulating discussions and comments.

REFERENCES

1. R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. **D11** (1975), 3432

2. H. Bergknoff and H.B. Thacker, Phys. Rev. Lett. **42** (1979), 135

3. T. Fujita and A. Ogura, Prog. Theor. Phys. **89** (1993), 23

4. K. Johnson, Nuovo Cimento, **20** (1961), 773

5. A. Ogura, T. Tomachi and T. Fujita, Ann. Phys. (N.Y.) **237** (1995), 12

6. W. Thirring, Ann. Phys. (N.Y) **3** (1958), 91

7. H.B. Thacker, Rev. Mod. Phys. **53** (1981), 253

8. B. Klaiber, in Lectures in Theoretical Physics, 1967, edited by A. Barut and W. Britten (Gordon and Breach, NY, 1968)

9. J. Schwinger, Phys. Rev. Lett. **3** (1959), 296

10. A. Ogura, Ph. D thesis, Nihon University (1994)

11. R.J. Baxter, Ann. Phys. (N.Y.) **70** (1972), 193

12. A. Luther, Phys. Rev. **B14** (1976), 2153

13. J.D. Johnson, S. Krinsky and B.M. McCoy, Phys. Rev. **A8** (1973), 2526

14. A.B.Zamolodchikov and A.B.Zamolodchikov, Ann. Phys. **120** (1979), 253

15. T. Fujita, C. Itoi and H. Mukaida, to be published.

16. T. Fujita and M. Hiramoto, to be published.

17. V.E. Korepin and L.D. Faddeev, Theor. Math. Phys. **25** (1975), 1039

Table 1

N=1601 L_0=100

	$\frac{g}{\pi} = 1$	$\frac{g}{\pi} = 1.25$	$\frac{g}{\pi} = 1.5$	$\frac{g}{\pi} = 1.7$
E_v	-9095.31	-6215.70	-4205.83	-2995.13
$E_{1p1h}^{(0)}$	-9089.43	-6210.69	-4201.76	-2991.83
$E_{1p1h}^{(1)}$	-9080.78	-6197.08	-4182.54	-2966.95
$E_{1p1h}^{(2)}$	-9080.72	-6197.02	-4182.48	-2966.89
$E_{1p1h}^{(3)}$	-9080.66	-6196.96	-4182.42	-2966.82
$E_{1p1h}^{(4)}$	-9080.59	-6196.90	-4182.35	-2966.76
$E_{1p1h}^{(5)}$	-9080.53	-6196.83	-4182.29	-2966.70
$E_{1p1h}^{(6)}$	-9080.47	-6196.77	-4182.23	-2966.64
$E_{2p2h}^{(1,-1)}$	-9066.23	-6178.55	-4159.13	-2938.79
$E_{2p2h}^{(2,-2)}$	-9066.10	-6178.43	-4158.99	-2938.66
$E_{2p2h}^{(3,-3)}$	-9065.97	-6178.30	-4158.86	-2938.52
$E_{2p2h}^{(4,-4)}$	-9065.85	-6178.17	-4158.74	-2938.38
$E_{2p2h}^{(5,-5)}$	-9065.72	-6178.04	-4158.61	-2938.25
$E_{2p2h}^{(6,-6)}$	-9065.59	-6177.91	-4158.49	-2938.12

We plot the calculated energies of E_v, $E_{1p1h}^{(n)}$ $(n = 0, 6)$ and $E_{2p2h}^{(n,-n)}$ $(n = 1, 6)$ for some values of the coupling constant $\frac{g}{\pi}$ with the fixed $L_0 = 100$. The number of particles here is $N = 1601$. Note that we put $m_0 = 1$ in our calculations.

Table 2

		Present Calculation	Fujita Ogura	Dashen et al.	Bergknoff Thacker
$\mathcal{M}$	$\frac{g}{\pi} = 0.8$ $(g_0 = 4.19)$	$1.01m$	$0.98m$	$0.83m$	$0.51m$
$\mathcal{M}$	$\frac{g}{\pi} = 1.0$ $(g_0 = 6.28)$	$0.75m$	$0.77m$	$0.62m$	$0.34m$
$\mathcal{M}$	$\frac{g}{\pi} = 1.25$ $(g_0 = 10.5)$	$0.47m$	$0.54m$	$0.41m$	$0.20m$

We plot the predicted values of the boson mass $\mathcal{M}$ by the present calculation, by the infinite momentum frame calculation (Fujita-Ogura), by the semiclassical method (Dashen et al.) and by the Bethe ansatz technique with *string* hypothesis (Bergknoff - Thacker).

Table 3

$N = 1601, \quad L_0 = 100$

	$\frac{g}{\pi} = 1$	$\frac{g}{\pi} = 1.25$	$\frac{g}{\pi} = 1.5$	$\frac{g}{\pi} = 1.7$
$E_{2p2h}^{(BB)*1}$	-9089.54	-6209.95	-4200.24	-2989.82
$E_{2p2h}^{(BB)*2}$	-9089.53	-6209.93	-4200.22	-2989.80
$E_{2p2h}^{(BB)*3}$	-9089.51	-6209.92	-4200.21	-2989.78
$E_{2p2h}^{(BB)*4}$	-9089.50	-6209.90	-4200.19	-2989.75
$E_{2p2h}^{(BB)*5}$	-9089.48	-6209.89	-4200.17	-2989.73
$E_{2p2h}^{(BB)*6}$	-9089.47	-6209.87	-4200.15	-2989.71

We plot the calculated values of the first six excitation energies of the boson boson states $(2p-2h)$ for four cases of the coupling constants with $N = 1601$ and $L_0 = 100$. Here, n of the $E_{2p2h}^{(BB)*n}$ denotes the $n-$th energy state from the lowest boson boson configurations.

Table 4a

$$[\frac{g}{\pi} = 1]$$

ρ_0	N	$\Delta E^{(BB)*1}_{2p2h}$	$\Delta E^{(0)}_{1p1h}$	$\Delta E^{(1)}_{1p1h}$	$\Delta E^{(1,-1)}_{2p2h}$
4	101	6.16	4.47	9.88	19.21
	201	6.06	4.45	9.99	19.71
	401	6.13	4.45	10.05	19.96
	1601	6.09	4.44	10.09	20.15
8	101	5.74	5.91	13.93	26.71
	201	5.76	5.89	14.25	27.90
	401	5.79	5.89	14.41	28.52
	1601	5.77	5.88	14.53	28.98
16	101	5.55	7.93	19.21	36.03
	201	5.59	7.91	19.94	38.61
	401	5.54	7.89	20.32	40.00
	1601	5.57	7.89	20.61	41.07
32	101	5.26	10.72	25.84	47.21
	201	5.30	10.69	27.40	52.14
	401	5.26	10.68	28.23	55.09
	1601	5.28	10.67	28.88	57.40
64	101	4.93	14.55	33.80	61.06
	201	4.97	14.52	36.95	68.58
	401	4.94	14.51	38.68	74.49
	1601	$*.$	14.48	40.03	79.28

We plot the calculated values of the lowest excitation energy of the boson boson states with respect to the vacuum energy as the function of N and ρ_0 for $\frac{g}{\pi} = 1$ case. Here, we also show the boson energy ($\Delta E^{(0)}_{1p1h}$) as well as the lowest excitation energy of the $1p-1h$ ($\Delta E^{(1)}_{1p1h}$) and $2p-2h$ ($\Delta E^{(1,-1)}_{2p2h}$ scattering states with respect to the vacuum energy. Here, $*$ shows that the present computor program could not find the solutions.

Table 4b

$$[\frac{g}{\pi} = 1.25]$$

ρ_0	N	$\Delta E^{(BB)*1}_{2p2h}$	$\Delta E^{(0)}_{1p1h}$	$\Delta E^{(1)}_{1p1h}$	$\Delta E^{(1,-1)}_{2p2h}$
4	101	5.70	3.84	11.79	23.19
	201	5.61	3.83	12.09	24.00
	401	5.61	3.82	12.25	24.41
	1601	5.61	3.82	12.37	24.71
8	101	5.68	5.02	17.12	33.32
	201	5.71	5.02	17.90	35.34
	401	5.72	5.01	18.30	36.39
	1601	5.75	5.01	18.62	37.18
16	101	5.68	6.75	24.02	45.68
	201	5.72	6.74	25.76	50.39
	401	5.74	6.73	26.69	52.86
	1601	5.73	6.72	27.42	54.73
32	101	5.57	9.21	32.44	59.79
	201	5.62	9.19	36.04	69.38
	401	5.58	9.18	38.05	74.82
	1601	$*$.	9.18	39.65	79.00
64	101	5.40	12.65	42.00	75.57
	201	$*$.	12.62	48.98	91.89
	401	$*$.	12.62	53.07	103.18
	1601	$*$.	12.60	$*$.	$*$.

The same as Table 4a. The coupling constant is $\frac{g}{\pi} = 1.25$.

<u>Table 4c</u>

$$[\frac{g}{\pi} = 1.5]$$

ρ_0	N	$\Delta E^{(BB)*1}_{2p2h}$	$\Delta E^{(0)}_{1p1h}$	$\Delta E^{(1)}_{1p1h}$	$\Delta E^{(1,-1)}_{2p2h}$
4	101	5.43	3.21	13.72	27.33
	201	5.31	3.21	14.29	28.57
	401	5.37	3.21	14.59	29.18
	1601	5.33	3.21	14.81	29.61
8	101	5.53	4.08	20.40	40.30
	201	5.55	4.08	21.88	43.67
	401	5.62	4.07	22.67	45.34
	1601	5.59	4.07	23.29	46.58
16	101	5.71	5.41	28.96	55.88
	201	5.74	5.40	32.29	63.97
	401	5.84	5.39	34.16	68.15
	1601	$*.$	5.39	35.65	71.28
32	101	5.74	7.34	39.02	72.31
	201	$*.$	7.33	45.82	89.22
	401	$*.$	$*.$	$*.$	$*.$
	1601	$*.$	$*.$	$*.$	$*.$
64	101	$*.$	$*.$	$*.$	$*.$
	201	$*.$	$*.$	$*.$	$*.$
	401	$*.$	$*.$	$*.$	$*.$
	1601	$*.$	$*.$	$*.$	$*.$

The same as Table 4a. The coupling constant is $\frac{g}{\pi} = 1.5$.

Table 4d

$$[\frac{g}{\pi} = 1.7]$$

ρ_0	N	$\Delta E_{2p2h}^{(BB)*1}$	$\Delta E_{1p1h}^{(0)}$	$\Delta E_{1p1h}^{(1)}$	$\Delta E_{2p2h}^{(1,-1)}$
4	101	5.10	2.73	15.46	31.19
	201	5.07	2.73	16.35	32.93
	401	4.99	2.73	16.82	33.78
	1601	4.98	2.72	17.17	34.38
8	101	5.24	3.31	23.37	46.94
	201	5.41	3.31	25.81	52.11
	401	5.30	3.31	27.14	54.65
	1601	5.31	3.30	28.18	56.49
16	101	5.53	4.24	33.29	64.98
	201	6.01	4.22	38.87	78.09
	401	*.	*.	*.	*.
	1601	*.	*.	*.	*.
32	101	*.	5.64	44.38	82.24
	201	*.	*.	*.	*.
	401	*.	*.	*.	*.
	1601	*.	*.	*.	*.
64	101	*.	*.	*.	*.
	201	*.	*.	*.	*.
	401	*.	*.	*.	*.
	1601	*.	*.	*.	*.

The same as Table 4a. The coupling constant is $\frac{g}{\pi} = 1.7$.

Figure captions:

Fig.1: In fig.1a, we show the configuration of the vacuum in the $E-k$ plane. Fig.1b shows the symmetric case of $1p-1h$ state while fig.1c the asymmetric case of $1p-1h$ state. Fig.1d shows the symmetric case of $2p-2h$ state.

Fig.2: The calculated spectrum for $\frac{g}{\pi} = 1.25$ with $L_0 = 110$ and $N = 1601$ is shown. The "boson" corresponds to $\Delta E^{(0)}_{1p1h}$ while all the other states are in the continuum. We identify the physical fermion mass such that the lowest energy of the $1p-1h$ continuum state is $2m$.

Fig.3: We show the excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ for the coupling constant $\frac{g}{\pi} = 0.8$ ($g_0 = 4.19$) (fig.3a), $\frac{g}{\pi} = 1.0$ ($g_0 = 6.28$) (fig.3b) and $\frac{g}{\pi} = 1.25$ ($g_0 = 10.5$) (fig.3c) cases as the function of ρ_0. Here, $E(0)$ and $E(1)$ denote $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$, respectively. The black dots denote the calculated results. The solid lines are straight lines for reference in log-log plot with the same slope α for $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$.

Fig.4: The same excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ are plotted as the function of ρ_0^{α}. The numbers on the lines denote the coupling constant $\frac{g}{\pi}$. The black circles are for $\Delta E^{(0)}_{1p1h}$ while the black squares for $\Delta E^{(1)}_{1p1h}$. The dashed lines are straight line for reference.

Fig.5: The boson mass is shown as the function of the coupling constant $\frac{g}{\pi}$. The black circles with error bars ($\Delta\mathcal{M}$) are the present calculation.

The solid line (FO) is the predicted boson mass by Fujita-Ogura, the dashed line (DHN) by Dashen et al. and the dashed-dotted line (BT) by Bergknoff and Thacker.

Fig.6: We show the calculated excitation energies $\Delta E^{(0)}_{1p1h}$ and $\Delta E^{(1)}_{1p1h}$ for higher coupling constants ($\frac{g}{\pi}$ = 1.5 and 1.7) as the function of ρ_0. The black dots denote for $\frac{g}{\pi} = 1.5$ while the white circles for $\frac{g}{\pi} = 1.7$. As can be seen, the slopes of the two excitation energies are not the same with each other.